JOURNAL OF APPROXIMATION THEORY **52**, 293–314 (1988)

Asymptotic Behaviour of the Ratio of Christoffel Functions for Weights W^2 and $W^2 g$

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Communicated by V. Totik

Received April 19, 1985; revised March 18, 1986

Under various assumptions on a weight $W^2 = \exp(-2Q)$ supported on $\mathbb{R}$, and on a positive function g, we establish the equality

$$\lim_{n \to \infty} \lambda_n(W^2 g, x)/\lambda_n(W^2, x) = g(x),$$

where $\lambda_n(W^2 g, x)$ and $\lambda_n(W^2, x)$ denote the Christoffel functions for the weights $W^2 g$ and W^2, respectively. Depending on the smoothness and rate of growth of g, we establish rates of convergence. The results apply to the weights $W_m(x) = \exp(-x^m/2)$, $m = 2, 4, 6, \ldots$. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let $d\alpha(x)$ be a nonnegative mass distribution on $\mathbb{R}$ with all moments finite. Let $\{p_n(d\alpha; x)\}$ be the orthonormal polynomials associated with $d\alpha$, so that

$$\int_{-\infty}^{\infty} p_n(d\alpha; x)\, p_m(d\alpha; x)\, d\alpha(x) = \delta_{mn}, \qquad m, n = 0, 1, 2, \ldots.$$

* Work of this author was supported in part by the Research Grants Division of the Council for Scientific and Industrial Research.

0021-9045/88 $3.00

In the theory of orthogonal polynomials, an important role is played by the Christoffel functions

$$\lambda_n(d\alpha, x) = 1 \Big/ \sum_{j=0}^{n-1} \{p_j(d\alpha; x)\}^2 \tag{1.1}$$

$$= \min \int_{-\infty}^{\infty} P^2(u)\, d\alpha(u)/P^2(x), \tag{1.2}$$

as is obvious from Freud [4] or Nevai [17]. Here the min is taken over all polynomials P of degree at most $n-1$.

In this note, we take up a thread of Freud [5, 6] and an idea of Nevai [17, Chap. 6] to establish the limit relation

$$\lim_{n \to \infty} \lambda_n(W^2 g, x)/\lambda_n(W^2, x) = g(x), \tag{1.3}$$

involving the Christoffel functions $\lambda_n(W^2 g, x)$ and $\lambda_n(W^2, x)$ associated with the weights $d\alpha_g(x) = W^2(x)\, g(x)\, dx$ and $d\alpha(x) = W^2(x)\, dx$, respectively. Apart from the intrinsic interest of (1.3), it is useful in deducing properties of the orthonormal polynomials and orthonormal expansions for $W^2 g$ from those of W^2 [17, Chaps. 6 and 8].

Freud's method for estimating Christoffel functions [5, 6, 13] consisted of approximating the weight on one side by polynomials, with equality at one point. We shall use a refinement of Freud's method in this note to establish (1.3), assuming g'' exists in a suitable interval—see Theorems 1.4 and 1.6. To compensate for this quite severe smoothness restriction on g, its growth may nevertheless be relatively rapid. By approximating continuous g by twice differentiable g, we obtain a convergence result without a rate of convergence— see Corollary 1.5.

As a contrast, we shall also obtain results (Theorem 1.2 and Corollary 1.3) assuming only continuity of g, and with rates of convergence involving the local modulus of continuity of g. The underlying feature here is the sequence of linear operators $\{G_n(d\alpha, f, x)\}$ introduced by Nevai in [17]. While Nevai considered weights on $[-1, 1]$, Knopfmacher [8] established the convergence of $G_n(W^2, f, x)$ and related operators for weights on the whole real line. As one might expect, weights on $\mathbb{R}$ introduce difficulties not encountered for weights on $[-1, 1]$. This explains the severe growth restriction on g in Theorem 1.2.

In order to state our main results, we need some notation. Throughout $C, C_1, C_2, C_3, \ldots$ denote positive constants independent of n, x, and u and (occasionally) independent of all polynomials of degree at most n. The same symbol does not necessarily indicate the same constant from line to line. We use the usual o, O, $\sim$ notation to compare functions and sequences. Thus, for example, $f(x) \sim g(x)$ if for some C_1 and C_2,

$$C_1 \leqslant f(x)/g(x) \leqslant C_2$$

for the relevant range of x.

DEFINITION 1.1. Let $W(x) = \exp(-Q(x))$, $x \in \mathbb{R}$, where Q is even, and twice differentiable in $(0, \infty)$. We say $W^2(x) = \exp(-2Q(x))$ is a *regular weight* if it satisfies

(a) *Explicit Assumptions*

$$Q(x) > 0 \quad \text{and} \quad Q'(x) > 0, \qquad x \in (0, \infty), \tag{1.4}$$

$$0 \leqslant Q''(x_1) \leqslant (1 + C_1)\, Q''(x_2), \qquad 0 < x_1 < x_2, \tag{1.5}$$

$$xQ''(x)/Q'(x) \leqslant C_2, \qquad x \in (0, \infty), \tag{1.6}$$

$$Q'(2x)/Q'(x) > 1 + C_3, \qquad x \text{ large enough.} \tag{1.7}$$

Associated with W are the numbers q_n, defined to be the positive root of the equation

$$q_n Q'(q_n) = n, \qquad n \geqslant 1. \tag{1.8}$$

(b) *Implicit Assumption*

There exist C_1 and C_2 such that

$$|p_n(W^2, x)|\, W(x) \leqslant C_1 q_n^{-1/2}, \qquad |x| \leqslant C_2 q_n,\, n \geqslant 1. \tag{1.9}$$

The explicit assumptions may be weakened substantially for the required properties of W^2 to hold. Further, (1.7) is implied by the other explicit assumptions on W^2. However, for brevity and ease of reference, we retain all the above restrictions on W^2. In any event, a proof of (1.9) has been published only for the weights

$$W_m(x) = \exp(-\tfrac{1}{2}x^m), \qquad m = 2, 4, 6, \ldots \tag{1.10}$$

(Bonan [3], Nevai [19]), though the bound (1.9) is known to be true for the slightly more general weight $W(x) = \exp(-Q(x))$, $Q(x)$ a polynomial of even degree with positive leading coefficient.

Of course (1.4) to (1.7) are valid for $W_\alpha(x) = \exp(-\frac{1}{2}|x|^\alpha)$, $\alpha \geqslant 2$, and one expects that (1.9) will ultimately be proved for $W_\alpha(x)$, any $\alpha > 1$. The only explicit assumption that is not satisfied by $W_\alpha(x)$, $1 < \alpha < 2$, is (1.5), but this can be dropped for the required properties of W^2 to hold (Levin and Lubinsky [9, 10]). In summary, at present only the weights (1.10) are known to be regular, but ultimately our results should apply to $W_\alpha(x)$, all $\alpha > 1$.

Let $f(u)$ be a function bounded near $\alpha \in \mathbb{R}$. The local modulus of continuity of f near x is

$$w_x(f; \varepsilon) = \sup\{|f(x) - f(y)|: y \in [x - \varepsilon, x + \varepsilon]\}, \qquad \varepsilon > 0,$$

while if f is uniformly continuous in $\mathbb{R}$, we set

$$w(f; \varepsilon) = \sup\{|f(x) - f(y)|: |y - x| \leqslant \varepsilon,\ x, y \in \mathbb{R}\}, \qquad \varepsilon > 0.$$

THEOREM 1.2. *Let $W^2(x)$ be a regular weight. Let g be positive, measurable, and finite valued in $\mathbb{R}$, and assume there exists $\beta \in [0, 2]$ such that*

$$\sup\{|g(x) - g(t)|/|x - t|^{\beta}: |x - t| \geqslant 1,\ x, t \in \mathbb{R}\} < \infty \tag{1.11}$$

and

$$\sup\{|g^{-1}(x) - g^{-1}(t)|/|x - t|^{\beta}: |x - t| \geqslant 1,\ x, t \in \mathbb{R}\} < \infty. \tag{1.12}$$

For $n = 1, 2, 3, ...,$ let

$$\begin{aligned} T_n &= 1, & \beta &< 1, \\ &= \log n, & \beta &= 1, \\ &= q_n^{\beta - 1}, & \beta &> 1, \end{aligned} \tag{1.13}$$

and assume

$$\lim_{n \to \infty} T_n q_n / n = 0. \tag{1.14}$$

Then there exists $C_1 > 0$ such that uniformly for $|x| < C_1 q_n$,

$$\begin{aligned} &\lambda_n(W^2 g, x)/(\lambda_n(W^2, x)\, g(x)) \\ &\quad = 1 + O\left((q_n/n)\left\{T_n(g(x) + g^{-1}(x)) \right.\right. \\ &\qquad \left.\left. + \int_{q_n/n}^{2} (g^{-1}(x)\, w_x(g; v) + g(x)\, w_x(g^{-1}; v))\, v^{-2}\, dv\right\}\right). \end{aligned} \tag{1.15}$$

In particular, uniformly in any compact interval in which g is continuous,

$$\lim_{n \to \infty} \lambda_n(W^2 g, x)/\lambda_n(W^2, x) = g(x). \tag{1.16}$$

Note that if $\beta < 2$, we have

$$T_n q_n / n = o(q_n^2/n) = o(1),$$

so that (1.14) is immediate. The case $\beta = 0$ is of special interest:

COROLLARY 1.3. *Let g and g^{-1} be positive, continuous, and bounded in $\mathbb{R}$. Then there exists C_1 such that uniformly for $|x| < C_1 q_n$,*

$$\lambda_n(W^2 g, x)/\lambda_n(W^2, x) = g(x) + O\left((q_n/n)\left\{1 + \int_{q_n/n}^{2} w_x(g; v)\, v^{-2}\, dv\right\}\right). \tag{1.17}$$

In particular, (1.16) *holds uniformly in any finite interval. If, further, g is uniformly continuous in $\mathbb{R}$, then uniformly for $|x| < C_1 q_n$,*

$$\lambda_n(W^2 g, x)/\lambda_n(W^2, x) = g(x) + o(1). \tag{1.18}$$

Together with the asymptotics of Nevai [18] for $\lambda_n(W_4^2, x)$ and of Sheen [20] for $\lambda_n(W_6^2, x)$, the above results yield asymptotics for $\lambda_n(W_4^2 g, x)$ and $\lambda_n(W_6^2 g, x)$—see Section 2. Using the results of Bauldry [2], one may obtain other asymptotics. Further results related to Theorem 1.2 and Corollary 1.3 appear in Knopfmacher [8].

Our next result has a "local" flavour: It assumes the existence of g'' only in a suitable interval.

THEOREM 1.4. *Let g be positive in $\mathbb{R}$, and let it have a positive lower bound and a finite upper bound in each finite interval. Let I be a closed bounded interval, and assume g' is absolutely continuous in I, while g'' is bounded in I. Let W^2 be a regular weight, and assume*

$$|\log g(x)| = o(Q(x)), \qquad |x| \to \infty. \tag{1.19}$$

Further, let

$$m_n = q_n^2(1 + \max\{|\log g(x)|/(1 + x^2) \colon |x| \leqslant 160 q_n\}), \tag{1.20}$$

and assume that

$$m_n = o(n), \qquad n \to \infty. \tag{1.21}$$

Then if J is a closed subinterval of I^0 (the interior of I), we have, uniformly in J,

$$\lambda_n(W^2 g, x)/(\lambda_n(W^2, x)\, g(x)) = 1 + O(m_n/n). \tag{1.22}$$

The number 160 in (1.20) can be replaced by a somewhat smaller (but fixed positive) number. However, such a replacement would not substantially strengthen the above result. The conditions (1.19) to (1.21) allow g to grow exponentially. For example, if $W(x) = W_4(x) = \exp(-1/2x^4)$, so that $q_n = (n/2)^{1/4}$, then (1.19) and (1.21) hold if

$$g(x) = \exp(o(x^4)), \qquad |x| \to \infty.$$

However, since (1.20) and (1.21) require that $q_n = o(\sqrt{n})$, Theorem 1.4 cannot be applied to $W(x) = W_2(x) = \exp(-1/2x^2)$. Some results for this weight are discussed in Section 3. By approximating continuous g by twice differentiable g, we can prove the following corollary:

COROLLARY 1.5. *Let g be positive in $\mathbb{R}$, and let it have a positive lower bound and a finite upper bound in each finite interval. Let I be a closed bounded interval, and assume that g is continuous in I. Let W^2 be a regular weight and assume that* (1.19), (1.20), *and* (1.21) *hold. Then if J is a closed subinterval of I^0, we have, uniformly in J,*

$$\lambda_n(W^2 g, x)/(\lambda_n(W^2, x)\, g(x)) = 1 + o(1).$$

Following is a "global" version of Theorem 1.4:

THEOREM 1.6. *Let g be positive in $\mathbb{R}$, and assume g' is absolutely continuous in $\mathbb{R}$, while g'' is bounded in each finite interval. Let W^2 be a regular weight and assume* (1.19) *holds. Let*

$$m_n = q_n^2(1 + \max\{|(\log g)''(x)| : |x| \leqslant 160 q_n\}), \tag{1.23}$$

$n = 1, 2, 3, \ldots$, *and assume that* (1.21) *holds. Then there exists $C_1 > 0$ such that* (1.22) *holds uniformly for $|x| \leqslant C_1 q_n$.*

One might expect that the range $|x| \leqslant C_1 q_n$ should really be $|x| \leqslant (1-\varepsilon)\, x_{n1}(W^2)$, where $\varepsilon > 0$ is arbitrary, and $x_{n1}(W^2)$ is the largest zero of $p_n(W^2; x)$. However, we cannot at present prove this even for $W = W_m$, $m = 8, 10, 12, \ldots$. The problem is that one needs the relation

$$\lambda_n(W^2, x) \sim (q_n/n)\, W^2(x), \qquad |x| \leqslant (1-\varepsilon)\, x_{n1}(W^2),$$

and at present this is known only for W_4 and W_6 (Nevai [18], Sheen [20]).

The paper is organized as follows: In Section 2, we prove Theorems 1.2 and Corollary 1.3, and state asymptotic results for $\lambda_n(W_4^2 g, x)$ and $\lambda_n(W_6^2 g, x)$. In Section 3, we prove Theorems 1.4 and 1.6 and discuss some related results for the Hermite weight.

2. PROOF OF THEOREM 1.2.

As discussed in the Introduction, the proof of Theorem 1.2 is based on a convergence theorem for certain linear operators. In order to introduce these operators, we need some notation. Throughout, W^2 is a regular weight, and $\{p_n(x)\} = \{p_n(W^2; x)\}$ denote the orthonormal polynomials associated with W^2. Further, $\gamma_n = \gamma_n(W^2) > 0$ denotes the leading coefficient of $p_n(W^2; x)$ and we let

$$K_n(x, y) = K_n(W^2; x, y) = \sum_{k=0}^{n-1} p_k(W^2; x)\, p_k(W^2; y) \tag{2.1}$$

$$= \frac{\gamma_{n-1}(W^2)}{\gamma_n(W^2)} \left\{ \frac{p_n(W^2; x)\, p_{n-1}(W^2; y) - p_n(W^2; y)\, p_{n-1}(W^2; x)}{x - y} \right\}. \tag{2.2}$$

In [7], Knopfmacher introduced and investigated a general class of continuous linear operators $\mathscr{G}_{n,p}$, $p \in [0, \infty)$, given by

$$\mathscr{G}_{n,p}[f](x) = \int_{-\infty}^{\infty} f(t)\, |K_n(x, t)|^p\, W^2(t)\, dt \Big/ \int_{-\infty}^{\infty} |K_n(x, t)|^p\, W^2(t)\, dt. \tag{2.3}$$

In the special case $p = 2$, it was shown in [7, Lemma 3.2(b)] that

$$\mathscr{G}_{n,2}[f](x) = \lambda_n(W^2, x) \int_{-\infty}^{\infty} f(t)(K_n(x, t))^2\, W^2(t)\, dt, \tag{2.4}$$

and this is the operator $\mathscr{G}_n(W^2, f, x)$ introduced by Nevai in [17]. In this section, we extend the convergence results in [7] for $G_{n,2}$ and then use these to prove Theorem 1.2.

LEMMA 2.1. *Let $W^2(x)$ satisfy the explicit assumptions in Definition* 1.1.

(i) *There exists C_1 such that uniformly for $|x| \leq C_1 q_n$,*

$$\lambda_n(W^2, x) \sim (q_n/n)\, W^2(x). \tag{2.5}$$

(ii) $\gamma_{n-1}/\gamma_n \leq C_2 q_n, \qquad n = 1, 2, 3, \ldots.$ (2.6)

(iii) *If in addition* (1.9) *holds, then there exists C_3 such that uniformly for $|x|$, $|t| \leq C_3 q_n$,*

$$|K_n(x, t)\, W(x)\, W(t)| \leq C_4/(|x - t| + q_n/n). \tag{2.7}$$

(iv) *There exists C such that*

$$C_6 n^C \leq q_n \leq C_5 n^{1/2}, \qquad n = 1, 2, 3, \ldots. \tag{2.8}$$

(v) $1 \leq q_{n+m}/q_n \leq 1 + C_6 m/n, \qquad n \geq m \geq 1.$ (2.9)

Proof. (i) See Freud [6, Lemmas 2.5 and 4.2] or Lubinsky [13].

(ii) See Freud [5, Lemma 2.7].

(iii) See Knopfmacher [7, Lemma 4.10].

(iv) For the upper bound on q_n, see Knopfmacher [7, Lemma 4.2g]. For the lower bound, see Lubinsky [12, Lemma 7(viii)].

(v) First, one may extend the definition of q_n from integral to nonintegral n: Define q_u to be the positive root of the equation

$$q_u Q'(q_u) = u, \qquad u \in (0, \infty).$$

It is shown in [12, Lemma 7(iii)] that

$$q_u'/q_u = 1/\{uT(q_u)\}, \qquad u > 0, \tag{2.10}$$

where

$$T(v) = 1 + vQ''(v)/Q'(v), \qquad v \in (0, \infty). \tag{2.11}$$

Then if $n \geqslant m \geqslant 1$,

$$\begin{aligned} 1 \leqslant q_{n+m}/q_n &= \exp\left(\int_n^{n+m} q_u'/q_u \, du\right) \\ &\leqslant \exp\left(C_7 \int_n^{n+m} du/u\right) \qquad \text{(by (1.6), (2.10), and (2.11))} \\ &= \exp(C_7 \log(1 + m/n)) \leqslant 1 + C_6 m/n. \quad \blacksquare \end{aligned}$$

THEOREM 2.2. (Convergence of $\mathscr{G}_{n,2}$). *Let f be a measurable function, finite valued in $\mathbb{R}$. Assume further, that there exists $\beta \in [0, 2]$ such that*

$$A = \sup\{|f(x) - f(t)|/|x - t|^\beta : |x - t| \geqslant 1,\ x, t \in \mathbb{R}\} < \infty. \tag{2.12}$$

Let $\{T_n\}$ be given by (1.13). *Then there exists C_1 such that uniformly for $|x| \leqslant C_1 q_n$,*

$$|\mathscr{G}_{n,2}[f](x) - f(x)| \leqslant C_2(q_n/n)\left\{T_n + \int_{q_n/n}^2 w_x(f; v)\, v^{-2}\, dv\right\}. \tag{2.13}$$

Proof. Now by (2.4) and as [7, Lemma 3.2(c)], $\mathscr{G}_{n,2}[1] \equiv 1$, we see that

$$\begin{aligned} |\mathscr{G}_{n,2}[f](x) - f(x)| &= \lambda_n(W^2, x) \\ &\quad \times \left|\int_{-\infty}^{\infty} (f(t) - f(x))\, K_n^2(x, t)\, W^2(t)\, dt\right| \\ &\leqslant \left\{\int_{|x-t| \leqslant 1} + \int_{\substack{|x-t|>1 \\ |t| \leqslant C_2 q_n}} + \int_{|t| \geqslant C_2 q_n}\right\} \\ &\quad \times \lambda_n(W^2, x) |f(t) - f(x)|\, K_n^2(x, t)\, W^2(t)\, dt. \end{aligned} \tag{2.14}$$

For convenience, let us denote the integrals in { } by Σ_1', Σ_2', and Σ_3', respectively. First, by Lemma 2.1(i) and (iii), there exists C_1 such that uniformly for $|x| \leqslant C_1 q_n$,

$$\Sigma_1' \leqslant C_3 \int_{|x-t| \leqslant 1} (q_n/n)|f(t)-f(x)|/(|x-t|+q_n/n)^2\,dt$$

$$\leqslant C_3 q_n/n \int_{q_n/n}^{1+q_n/n} w_x(f; v)\, v^{-2}\,dv, \tag{2.15}$$

as $w_x(f; v)$ is nondecreasing. Next, if C_2 is small enough, and $|x| < C_1 q_n$, Lemma 2.1(i) and (iii) show that

$$\Sigma_2' \leqslant C_3(q_n/n) \int_{\substack{|x-t|>1\\ |t| \leqslant C_2 q_n}} |f(x)-f(t)|/|x-t|^2\,dt$$

$$\leqslant C_3 A(q_n/n) \int_{\dots} |x-t|^{\beta-2}\,dt$$

$$\leqslant C(q_n/n)\, T_n, \tag{2.16}$$

by (2.12), and the definition (1.13) of $\{T_n\}$, and since Lemma 2.1(iv) shows that $\log q_n \sim \log n$. Finally, we must estimate Σ_3'. First note that from (1.9), (2.2), and (2.6), we have for $|x| \leqslant C_1 q_n$, and for all $t \in \mathbb{R}$,

$$|K_n(x,t)|^2 \leqslant (C_4 q_n^{1/2} W^{-1}(x)\{|p_n(t)| + |p_{n-1}(t)|\}/|x-t|)^2$$

$$\leqslant C_5 q_n\, W^{-2}(x)\{p_n^2(t) + p_{n-1}^2(t)\}/(x-t)^2. \tag{2.17}$$

Thus for $|x| \leqslant C_1 q_n$, with $C_1 < C_2$,

$$\Sigma_3' \leqslant C(q_n^2/n) \int_{|t| > C_2 q_n} |f(t)-f(x)|$$

$$\times \{p_n^2(t) + p_{n-1}^2(t)\}\, W^2(t)/(x-t)^2\,dt$$

$$\leqslant C(q_n^2/n)\, A q_n^{\beta-2} \int_{-\infty}^{\infty} \{p_n^2(t) + p_{n-1}^2(t)\}\, W^2(t)\,dt$$

$$= 2C(q_n/n)\, A q_n^{\beta-1}, \tag{2.18}$$

by (2.12) and orthonormality. Since (1.13) shows that

$$T_n \geqslant q_n^{\beta-1}, \qquad n \text{ large enough},$$

(2.14) to (2.18) yield (2.13). ∎

We note that the rate of convergence of $\mathscr{G}_{n,2}$ above is, in general, best possible (Knopfmacher [8, Theorem 5.5]). For the purposes of the following lemma, a function $w(x)$ is a weight, if it is nonnegative and measurable, and $\{x: w(x) > 0\}$ has positive measure, while all moments of w are finite.

LEMMA 2.3. *Let $g \geqslant 0$. Let W^2 be a regular weight. If $W^2 g$ is a weight, then*

$$\lambda_n(W^2 g, x)/\lambda_n(W^2, x) \leqslant \mathscr{G}_{n,2}[g](x), \qquad x \in \mathbb{R}. \tag{2.19}$$

If also $W^2 g^{-1}$ is a weight, then

$$\lambda_n(W^2 g, x)/\lambda_n(W^2, x) \geqslant \{\mathscr{G}_{n,2}[g^{-1}](x)\}^{-1}, \qquad x \in \mathbb{R}. \tag{2.20}$$

Proof. See Theorem 6.2.3 in Nevai [17, p. 76]. ∎

Proof of Theorem 1.2. We shall apply Theorem 2.2 to g and g^{-1}. In view of (1.11) and (1.12), both $f = g$ and $f = g^{-1}$ satisfy (2.12). By Theorem 2.2 and (2.19),

$$\begin{aligned}&\lambda_n(W^2 g, x)/\lambda_n(W^2, x)\\ &\quad \leqslant g(x) + C_2(q_n/n)\left\{T_n + \int_{q_n/n}^{2} w_x(g; v)\, v^{-2}\, dv\right\},\end{aligned} \tag{2.21}$$

uniformly for $|x| \leqslant C_1 q_n$. Further, by Theorem 2.2 and (2.20),

$$\begin{aligned}&\lambda_n(W^2 g, x)/\lambda_n(W^2, x)\\ &\quad \geqslant \left\{g^{-1}(x) + C_2(q_n/n)\left\{T_n + \int_{q_n/n}^{2} w_x(g^{-1}; v)\, v^{-2}\, dv\right\}\right\}^{-1}\\ &\quad = g(x)\left\{1 + O\left(g(x)(q_n/n)\left\{T_n + \int_{q_n/n}^{2} w_x(g^{-1}; v)\, v^{-2}\, dv\right\}\right)\right\}.\end{aligned} \tag{2.22}$$

This last step is valid even if the order term does not approach 0 uniformly for $|x| \leqslant C_1 q_n$, as $n \to \infty$, since then the right-hand side of (2.22) may be 0. The uniformity of (2.22) for $|x| \leqslant C_1 q_n$ merely reflects the fact that the constant in the order term is independent of n and x. Now (1.15) follows easily from (2.21) and (2.22). Finally, if g is continuous in a compact interval I, then we see that uniformly for $x \in I$,

$$\lim_{v \to 0+} \left(g^{-1}(x)\, w_x(g; v) + g(x)\, w_x(g^{-1}; v)\right) = 0,$$

and then (1.15) implies (1.16). ∎

Proof of Corollary 1.3. As g and g^{-1} are bounded, (1.11) and (1.12) hold with $\beta = 0$. Further, for $x, y \in \mathbb{R}$,

$$\begin{aligned}|g^{-1}(x) - g^{-1}(y)| &= |g(y) - g(x)|/\{g(x)\, g(y)\}\\ &\leqslant C|g(y) - g(x)|.\end{aligned}$$

Thus $w_x(g^{-1}; v) \leqslant C w_x(g; v)$, and (1.17) follows from (1.15). If also g is uniformly continuous in $\mathbb{R}$, then we may replace $w_x(g; v)$ by $w(g; v)$ in (1.17). Then (1.18) follows easily. ∎

For $\exp(-x^4/2)$ and $\exp(-x^6/12)$, we can use results of Nevai [18] and Sheen [20] to obtain precise asymptotics for $\lambda_n(W^2 g, x)$:

THEOREM 2.4. *Let $g(x)$ be positive, measurable, and finite valued in $\mathbb{R}$ and assume there exists $\beta \in [0, 2]$ such that* (1.11) *and* (1.12) *are true. Let*

$$\phi(x) = g(x) + g^{-1}(x), \qquad x \in \mathbb{R}, \tag{2.23}$$

and

$$\psi(x, v) = g(x)\, w_x(g^{-1}; v) + g^{-1}(x)\, w_x(g; v), \qquad v > 0,\ x \in \mathbb{R}. \tag{2.24}$$

(i) *Let $W(x) = W_4(x) = \exp(-x^4/2)$, $x \in \mathbb{R}$, and let*

$$h_1(\theta) = 2(12)^{1/4}\, (3\pi)^{-1} \sin\theta \{1 + 2\cos^2\theta\}, \qquad \theta \in [0, \pi]. \tag{2.25}$$

Then there exists $0 < \varepsilon_0 < \pi/2$ such that uniformly for $\theta \in [\varepsilon_0, \pi - \varepsilon_0]$ and $x = (4n/3)^{1/4} \cos\theta$,

$$\lambda_n(W_4^2 g, x) / \{(W_4^2 g)(x)\, n^{-3/4} h_1^{-1}(\theta)\}$$
$$= 1 + O\left(n^{-3/4} \left\{T_n \phi(x) + \int_{n^{-3/4}}^{1} \psi(x, v)\, v^{-2}\, dv\right\}\right). \tag{2.26}$$

Here $\{T_n\}$ is given by (1.13) *with $q_n = (n/2)^{1/4}$, $n = 1, 2, \ldots$.*

(ii) *Let $W(x) = W_6^{1/6}(x) = \exp(-x^6/12)$, $x \in \mathbb{R}$, and let*

$$h_2(\theta) = 10^{-5/6} \pi^{-1} \sin\theta (16\cos^4\theta + 8\cos^2\theta + 6), \qquad \theta \in [0, \pi]. \tag{2.27}$$

Then there exists $0 < \varepsilon_0 < \pi/2$ such that uniformly for $\theta \in [\varepsilon_0, \pi - \varepsilon_0]$ and $x = (32n/5)^{1/6} \cos\theta$,

$$\lambda_n(W^2 g, x) / \{(W^2 g)(x)\, n^{-5/6} h_2^{-1}(\theta)\}$$
$$= 1 + O\left(n^{-5/6} \left\{T_n \phi(x) + \int_{n^{-5/6}}^{1} \psi(x, v)\, v^{-2}\, dv\right\}\right). \tag{2.28}$$

Here $\{T_n\}$ is given by (1.13) *with $q_n = (2n)^{1/6}$, $n = 1, 2, \ldots$.*

Proof. (i) Nevai [18, Theorem 2] has shown that

$$n^{-3/4} \lambda_n^{-1}(W_4^2, x)\, W_4^2(x) = h_1(\theta) + O(n^{-1}),$$

uniformly for $x=(4n/3)^{1/4}\cos\theta$, $\varepsilon\leqslant\theta\leqslant\pi-\varepsilon$, and any fixed $0<\varepsilon<\pi/2$. Then Theorem 1.2 yields the result, as $\phi(x)\geqslant 1$, so $n^{-1}=O((q_n/n)\,T_n\phi(x))$.

(ii) This is an application of Theorem 2 in Sheen [20, Chap. 3], which states that

$$n^{-5/6}\lambda_n^{-1}(W^2;x)\,W^2(x)=h_2(\theta)+O(n^{-1}),$$

uniformly for $x=(32n/5)^{1/6}\cos\theta$, $\varepsilon\leqslant\theta\leqslant\pi-\varepsilon$, and any fixed $0<\varepsilon<\pi/2$. ∎

We remark that in the above result, one may replace "there exists $0<\varepsilon_0<\pi/2$ such that" by "for any $0<\varepsilon_0<\pi/2$." The reason for this is that the relevant bounds on the orthonormal polynomials and Christoffel functions hold uniformly for $\theta\in[\varepsilon,\pi-\varepsilon]$, for any $0<\varepsilon<\pi/2$. All that one needs to do is keep track of the constants in Theorem 2.2.

Using the results of Bauldry [2], one may also obtain asymptotics for $\lambda_n(W^2g,x)$, where $W=\exp(-Q)$ and Q is an arbitrary polynomial of degree 4 with positive leading coefficient.

3. Proof of Theorems 1.4 and 1.6

Freud's method of estimating Christoffel functions consists of approximating a weight on one side by polynomials, with equality at one point. We proceed with a modified form of the construction.

Lemma 3.1. *Let*

$$S_l(x)=\sum_{j=0}^{l} x^j/j!,\qquad l=0,1,2,\ldots. \tag{3.1}$$

Let $\theta_0=0.873$. *Then*

$$\max_{|x|\leqslant l/4}|1-S_l(x)/e^x|\leqslant C\theta_0^l,\qquad l=0,1,2,3,\ldots. \tag{3.2}$$

Proof. We see that for large enough l,

$$\begin{aligned}\max_{|x|\leqslant l/4}|1-S_l(x)/e^x| &= \max_{|x|\leqslant l/4}\left|e^{-x}\sum_{j=l+1}^{\infty}x^j/j!\right|\\ &\leqslant \exp(l/4)\sum_{j=l+1}^{\infty}(l/4)^j/j!\\ &\leqslant \exp(l/4)(l/4)^{l+1}/(l+1)!\\ &\quad\times\sum_{j=l+1}^{\infty}(l/4)^{j-l-1}/\{(l+2)(l+3)\ldots j\}\end{aligned}$$

$$\leqslant \exp(l/4)(le/(4(l+1)))^{l+1}$$
$$\times l^{-1/2}(1-\tfrac{1}{4})^{-1} \qquad \text{(by Stirling's formula)}$$
$$\leqslant \exp(l\{\tfrac{1}{4}-\log 4+1\}) \leqslant \theta_0^l. \quad \blacksquare$$

LEMMA 3.2. *Assume the hypotheses of Theorem 1.4, and let J be a closed subinterval of I^0. Then for each $x \in J$, there exists polynomials $V_n(u)$ and $U_n(u)$ (depending on x) of degree at most $C_1 m_n$, such that for $n = 1, 2, 3, \ldots$,*

$$V_n(x) = g^{1/2}(x) = U_n^{-1}(x), \tag{3.3}$$

and

$$V_n(u)(1 - C_2\theta^{m_n}) \leqslant g^{1/2}(u) \leqslant U_n^{-1}(u)(1 + C_2\theta^{m_n}), \tag{3.4}$$

uniformly for $|u| \leqslant 160q_n$. The constants C_1 and C_2 are independent of $x \in J$, $|u| \leqslant 160q_n$, and $n \geqslant 1$, while $\theta = 0.873$ as in Lemma 3.1.

Proof. Let

$$h(x) = \log g^{1/2}(x), \qquad x \in \mathbb{R}, \tag{3.5}$$

and for $x \in J$, let

$$V_n(u) = g^{1/2}(x)\, S_{l(n)}(h'(x)(u-x) - A_n(u-x)^2), \qquad u \in \mathbb{R}, \tag{3.6}$$

and

$$U_n(u) = g^{-1/2}(x)\, S_{l(n)}(-h'(x)(u-x) - A_n(u-x)^2), \qquad u \in \mathbb{R}, \tag{3.7}$$

where $S_{l(n)}$ is given by (3.1), and $l(n)$ and $A_n \geqslant 1$ will be chosen below. First, (3.3) follows directly from (3.6) and (3.7). Next, for $x \in J$ and $|u| \leqslant 160q_n$,

$$|h'(x)(u-x) \pm A_n(u-x)^2| \leqslant C_2 q_n + A_n(Cq_n)^2 \leqslant C_3 A_n q_n^2.$$

Thus if

$$l(n) \geqslant 4C_3 A_n q_n^2, \tag{3.8}$$

Lemma 3.1 and (3.5) show that uniformly for $x \in J$ and $|u| \leqslant 160q_n$,

$$\begin{aligned} g^{1/2}(u)/V_n(u) = \exp(&h(u) - h(x) - h'(x)(u-x) \\ &+ A_n(u-x)^2)(1 + O(\theta^{l(n)})) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} g^{1/2}(u)\, U_n(u) = \exp(&h(u) - h(x) - h'(x)(u-x) \\ &- A_n(u-x)^2)(1 + O(\theta^{l(n)})). \end{aligned} \tag{3.10}$$

Then (3.4) follows, and the proof of the lemma is complete, provided we can choose A_n and $l(n)$ (independent of $x \in J$) to satisfy (3.8) and

$$m_n \leqslant l(n) \leqslant Cm_n, \tag{3.11}$$

and

$$|h(u) - h(x) - h'(x)(u-x)| \leqslant A_n(u-x)^2, \qquad |u| \leqslant 160q_n,\ x \in J. \tag{3.12}$$

First, if $x \in J$ and $u \in I$, the existence and boundedness of g'' (and hence h'') ensure the existence of ξ between u and x, such that

$$h(u) - h(x) - h'(u)(u-x) = h''(\xi)(u-x)^2/2,$$

and then (3.12) is true for $x \in J$ and $u \in I$, provided

$$A_n \geqslant \max\{|h''(v)| : v \in I\}. \tag{3.13}$$

Next, we consider $x \in J$ and $u \notin I$. Since the distance d, say, from the endpoints of J to those of I is positive, it is not difficult to see that

$$(1+u^2)/(u-x)^2 \leqslant C, \qquad u \notin I,\ x \in J. \tag{3.14}$$

Then for $x \in J$ and $u \notin I$, but $|u| \leqslant 160q_n$,

$$\begin{aligned}
&|h(u) - h(x) - h'(x)(u-x)|/(u-x)^2 \\
&\quad \leqslant C|h(u)|/(1+u^2) + |h(x)|\, d^{-2} + |h'(x)|\, d^{-1} \\
&\qquad \text{(by (3.14) and choice of } d\text{)} \\
&\quad \leqslant (C/2)|\log g(u)|/(1+u^2) + C_4 d^{-2} + C_5 d^{-2} \\
&\quad \leqslant A_n
\end{aligned}$$

by (3.5) and provided

$$A_n \geqslant (C/2) \max\{|\log g(u)|/(1+u^2) : |u| \leqslant 160q_n\} + C_4 d^{-2} + C_5 d^{-1}. \tag{3.15}$$

Thus we can choose A_n to satisfy (3.13) and (3.15), and in view of the definition (1.20) of m_n can then choose l_n to satisfy (3.8) and (3.11). ▮

Having completed the construction of the polynomials, we shall need two more lemmas:

LEMMA 3.3. *Let W^2 be a regular weight. Let $\{m(n)\}$ be a given sequence of positive integers with*

$$m(n) = o(n), \qquad n \to \infty. \tag{3.16}$$

Then there exists C_1 such that uniformly for $|x| \leqslant C_1 q_n$,

$$\lambda_{n \pm m(n)}(W^2, x)/\lambda_n(W^2, x) = 1 + O(m(n)/n), \qquad n \to \infty. \tag{3.17}$$

Proof. By (1.1),

$$W^2(x)\{\lambda_{n\pm m(n)}^{-1}(W^2, x) - \lambda_n^{-1}(W^2, x)\} = \pm \sum_{j=n}^{n \pm m(n)} p_j^2(W^2; x)\, W^2(x)$$
$$= O\left(\sum_{j=n}^{n \pm m(n)} q_j^{-1}\right),$$

uniformly for $|x| \leqslant C_1 q_{n-m(n)}$, by (1.9). In view of Lemma 2.1(i) and (v), we deduce that uniformly for $|x| \leqslant C_1 q_n$ (some suitable C_1),

$$|1 - \lambda_{n\pm m(n)}(W^2, x)/\lambda_n(W^2, x)| = O(m(n)\, q_n^{-1} \lambda_{n\pm m(n)}(W^2, x)/W^2(x))$$
$$= O(m(n)/n). \quad \blacksquare$$

Finally, we need an infinite–finite range inequality. Unfortunately, the inequality given in Lubinsky [11, Theorem A] is not sufficiently general for our purposes, so we prove a suitable inequality using the method of [11]. More precise inequalities may be proved using the methods of potential theory (Lubinsky [14], Mhaskar and Saff [15, 16]).

LEMMA 3.4. *Let $W(u) = \exp(-Q(u))$, where Q is even, and twice differentiable in $(0, \infty)$. Assume further that*

$$Q''(x) \geqslant 0, \qquad x \in (0, \infty), \tag{3.18}$$

and that Q satisfies (1.4) and (1.6). Let $G(u)$ be a function positive, measurable, and bounded in each finite interval, with

$$\log G(u) = o(Q(u)), \qquad |u| \to \infty. \tag{3.19}$$

Let $0 < p_1 < \infty$. Then there exists $0 < \theta_1 < 1$ and $C > 0$ such that for every polynomial P of degree at most n, and for all $p \in [p_1, \infty]$,

$$\|PWG\|_{L_p(\mathbb{R})} \leqslant (1 + C\theta_1^n)^{1/p}\, \|PWG\|_{L_p(-160q_n,\, 160q_n)}. \tag{3.20}$$

Proof. For $n = 1, 2, 3, \ldots$, let ξ_n denote a real number such that

$$(GW)(\xi_n)|\xi_n|^n \geqslant (1 + 1/n)^{-1}\, \|(GW)(x)\, x^n\|_{L_\infty(\mathbb{R})}. \tag{3.21}$$

It is not difficult to see that

$$\lim_{n\to\infty} |\xi_n| = \infty.$$

We claim that for large enough n,

$$q_{n/2}/2 \leqslant |\xi_n| \leqslant 2q_{2n}. \tag{3.22}$$

In fact, (3.22) can be improved substantially, but is sufficient for our purposes. To prove (3.22), note first that given $\varepsilon > 0$, (3.19) shows that there exists $u_0 > 0$ with

$$W(u)^{\varepsilon} \leqslant G(u) \leqslant W(u)^{-\varepsilon}, \qquad |u| \geqslant u_0. \tag{3.23}$$

As is well known, the root q_{2n} of (1.8) satisfies

$$W(u)\, u^{2n} \leqslant W(q_{2n})\, q_{2n}^{2n}, \qquad u \in \mathbb{R}. \tag{3.24}$$

Then if n is large enough, (3.23) and (3.24) show that for $|u| \geqslant 2q_{2n}$,

$$\begin{aligned} G(u)\, W(u)\, u^n/\{G(q_{2n})\, W(q_{2n})\, q_{2n}^n\} &\leqslant W(u)^{1-\varepsilon}\, W(q_{2n})^{-1-\varepsilon}\, (u/q_{2n})^n \\ &\leqslant W(q_{2n})^{-2\varepsilon}\, (u/q_{2n})^{n-2n(1-\varepsilon)} \\ &\leqslant e^{2\varepsilon C n} 2^{-n(1-2\varepsilon)} < \tfrac{1}{2}, \end{aligned}$$

if n is large enough, as $\varepsilon > 0$ is arbitrary. Here we have used the fact [12, Lemma 7(vii)] that

$$\limsup_{n\to\infty} Q(q_{2n})/n < C < \infty. \tag{3.25}$$

It follows that if n is large enough and $|u| \geqslant 2q_{2n}$, then $u \neq \xi_n$, and so the right inequality in (3.22) is true. Next, if $u_0 \leqslant |u| \leqslant q_{n/2}/2$,

$$\begin{aligned} &G(u)\, W(u)\, u^n/\{G(q_{n/2})\, W(q_{n/2})\, q_{n/2}^n\} \\ &\leqslant W(u)^{1-\varepsilon}\, W(q_{n/2})^{1-\varepsilon}\, (u/q_{n/2})^n \\ &\leqslant W(q_{n/2})^{-2\varepsilon}\, (u/q_{n/2})^{n-(n/2)(1-\varepsilon)} \\ &\leqslant e^{\varepsilon C n} 2^{-(n/2)(1+\varepsilon)} < \tfrac{1}{2} \end{aligned}$$

by the obvious analogues of (3.24) and (3.25). It follows that $u \neq \xi_n$ and so the left inequality in (3.22) is also valid.

Now let P be a polynomial of degree $m \leqslant n$, not identically zero, and write $H = |\xi_{2n}|$ and

$$P(u) = c \prod_{i=1}^{m} (u - u_i),$$

with $c \neq 0$ and

$$|u_i| \leqslant 20H, \qquad 1 \leqslant i \leqslant j,$$
$$|u_i| > 20H, \qquad j < i \leqslant m.$$

Let $0 < r < 1/(4e)$. Then if $|x| \geqslant 20H$, $|u| \leqslant H$, and $j < i \leqslant m$,

$$\left|\frac{x-u_i}{u-u_i}\right| \leqslant \frac{1+|x|/(20H)}{1-|u|/(20H)} \leqslant \frac{2|x|/(20H)}{1-1/(20)} \leqslant \frac{2|x|}{19H},$$

while if $1 \leqslant i \leqslant j$,

$$\left|\frac{x-u_i}{u-u_i}\right| \leqslant \frac{2|x|}{|u-u_i|}.$$

Thus if $|x| \geqslant 20H$, $|u| \leqslant H$,

$$\begin{aligned}|P(x)/P(u)| &\leqslant (2|x|/(19H))^{m-j}\,(2|x|)^j \Big/ \prod_{i=1}^{j} |u-u_i| \\ &\leqslant (2|x|/(rH))^m \leqslant (2|x|/(rH))^n,\end{aligned}$$

provided also that $u \notin \mathscr{S}$, where $\mathscr{S}$ is open and has linear measure at most $4erH$. Here we have used Cartan's lemma on small values of polynomials (see, for example, Baker [1, p. 174]). Then for $|x| \geqslant 20H$, $|u| \leqslant H$, $u \notin \mathscr{S}$,

$$\begin{aligned}|P(x)\,W(x)\,G(x)| &\leqslant \left(\frac{2|x|}{rH}\right)^n |P(u)|\,\frac{|W(x)\,G(x)\,x^{2n}|}{W(\xi_{2n})\,G(\xi_{2n})\,\xi_{2n}^{2n}} \\ &\quad \times (WG)(\xi_{2n})\left(\frac{\xi_{2n}}{x}\right)^{2n} \\ &\leqslant \left(\frac{2H}{r|x|}\right)^n |P(u)|\,2(WG)(\xi_{2n}),\end{aligned} \qquad (3.26)$$

by (3.21). Now by (3.18), Q' is nondecreasing, so

$$Q(2v)-Q(v)=\int_v^{2v} Q'(u)\,du \geqslant \int_o^v Q'(u)\,du = Q(v)-Q(0),$$

and consequently

$$Q(2v) \geqslant 2Q(v)(1+o(1)), \qquad |v| \to \infty.$$

Then if $u_0 \leqslant |u| \leqslant H/2$, (1.4) and (3.23) show that

$$\begin{aligned}(WG)(\xi_{2n})/(WG)(u) &\leqslant \exp(Q(u)(1+\varepsilon) - Q(\xi_{2n})(1-\varepsilon)) \\ &\leqslant \exp(Q(\xi_{2n}/2)(1+\varepsilon) - Q(\xi_{2n}/2)\, 2(1-\varepsilon)(1+o(1))) \\ &\leqslant 1, \qquad (3.27)\end{aligned}$$

if n is large enough. Let

$$\mathcal{M} = [-H/2, H/2] \backslash (\mathcal{S} \cup [-u_0, u_0]).$$

Then

$$\begin{aligned}\text{meas}\, \mathcal{M} &\geqslant H(1-4er) - 2u_0 \\ &\geqslant C_1 H, \qquad (3.28)\end{aligned}$$

if n is large enough. Then if $|x| \geqslant 20H$ and $u \in \mathcal{M}$, (3.26) and (3.27) show that

$$|P(x)\, W(x)\, G(x)| \leqslant (2H/(r|x|))^n\, 2|P(u)\, W(u)\, G(u)|,$$

and by (3.28), for any $p > 0$,

$$|P(x)\, W(x)\, G(x)|^p \leqslant (2H/(r|x|))^{np}\, 2^p \int_{\mathcal{M}} |P(u)\, W(u)\, G(u)|^p\, du/(C_1 H),$$

and so integrating with respect to $|x|$ from $20H$ to ∞,

$$\begin{aligned}&\left(\int_{|x| \geqslant 20H} |P(x)\, W(x)\, G(x)|^p\, dx\right)^{1/p} \\ &\quad \leqslant 2\left(\int_{-H/2}^{H/2} |P(u)\, W(u)\, G(u)|^p\, du\right)^{1/p} (10r)^{-n}\, ((40/C_1)/(np-1))^{1/p}.\end{aligned}$$

Let $\theta_1 = (10r)^{-1} < 1$ if $1/10 < r < 1/(4e)$. Now by (3.22),

$$H = |\xi_{2n}| \leqslant 2q_{4n} \leqslant 8q_n,$$

since (1.8) shows that

$$q_{4n}/q_n = 4Q'(q_n)/Q'(q_{4n}) \leqslant 4.$$

Hence for $p_1 \leqslant p < \infty$ and $n \geqslant n_1(p_1)$, we have

$$\|PWG\|_{L_p(|x| \geqslant 160q_n)} \leqslant 2\theta_1^n (20/C_1) \|PWG\|_{L_p[-4q_n, 4q_n]}$$

for all polynomials P of degree $\leqslant n$. Since the constants in this last inequality are independent of p, we can let $p \to \infty$, so that it also holds for $p = \infty$. Then (3.20) follows, possibly with a larger $\theta_1 < 1$. ∎

We remark that when $G \equiv 1$, Theorem A in [11, p. 264] shows that we may replace $160q_n$ in (3.20) by $11q_{2n}$, and hence by $22q_n$. We shall need this in the

Proof of Theorem 1.4. Let J be a closed subinterval of I^0, and let V_n and U_n be the polynomials constructed in Lemma 3.2. We may assume they have degree at most m_n, so that $C_1 = 1$. Now for $x \in J$, (3.3) and (3.4) show that

$$\begin{aligned}\lambda_n(W^2 g, x) &\geqslant \inf_{\deg(P) \leqslant n-1} \int_{-160q_n}^{160q_n} (PW)^2(u)\, g(u)\, du/P^2(x)\\ &\geqslant (1+O(\theta^{m_n}))\, g(x)\\ &\quad\times \inf_{\deg(P) \leqslant n-1} \int_{-160q_n}^{160q_n} (PV_n W)^2(u)\, du/(PV_n)^2(x)\\ &\geqslant (1+O(\theta^{m_n}))\, g(x)\\ &\quad\times \inf_{\deg(P) \leqslant n+m_n-1} \int_{-160q_n}^{160q_n} (PW)^2(u)\, du/P^2(x)\\ &\geqslant (1+O(\theta^{m_n}))(1+O(\theta_1^n))\, g(x)\\ &\quad\times \inf_{\deg(P) \leqslant n+m_n-1} \int_{-\infty}^{\infty} (PW)^2(u)\, du/P^2(x),\end{aligned}$$

since $n + m_n = n(1+o(1))$ for large n, and $160q_n \geqslant 22q_{n+m_n}$ for large enough n, by Lemma 2.1(v). Then, by Lemma 3.3, uniformly for $x \in J$, and for n large enough,

$$\begin{aligned}\lambda_n(W^2 g, x) &\geqslant (1+O(n^{-1}))\, g(x)\, \lambda_{n+m_n}(W^2, x)\\ &\geqslant (1+O(m_n/n))\, g(x)\, \lambda_n(W^2, x).\end{aligned} \tag{3.29}$$

Next, by Lemma 3.4 with $G = g^{1/2}$, and by (3.3) and (3.4),

$$\begin{aligned}\lambda_n(W^2 g, x) &\leqslant (1+O(\theta_1^n))\\ &\quad\times \inf_{\deg(P) \leqslant n-1} \int_{-160q_n}^{160q_n} (PW)^2(u)\, g(u)\, du/P^2(x)\\ &\leqslant (1+O(\theta_1^n))(1+O(\theta^{m_n}))\, g(x)\\ &\quad\times \inf_{\deg(P_1) \leqslant n-m_n-1} \int_{-160q_n}^{160q_n} (P_1 W)^2(u)\, du/P_1^2(x),\end{aligned}$$

where we have set $P=P_1U_n$. We deduce that uniformly for $x\in J$, and n large enough,

$$\lambda_n(W^2g, x)\leqslant(1+O(n^{-1}))\, g(x)\, \lambda_{n-m_n}(W^2, x)$$
$$\leqslant(1+O(m_n/n))\, g(x)\, \lambda_n(W^2, x),$$

by Lemma 3.3. ∎

Proof of Corollary 1.5. Let J be a closed subinterval of I^0, and let $\varepsilon>0$. It is easy to see that we can find a function $g_1(x)$, twice continuously differentiable and positive in I, such that

$$1-\varepsilon\leqslant g(x)/g_1(x)\leqslant 1+\varepsilon \tag{3.30}$$

and such that $g_1(x)=g(x)$, $x\notin I$. In fact, Weierstrass' theorem shows that we may choose g_1 to be a polynomial in I. Then applying Theorem 1.4 to g_1, we see there exists n_1 such that for $n\geqslant n_1$, and for all $x\in J$,

$$1-\varepsilon\leqslant\lambda_n(W^2g_1, x)/(\lambda_n(W^2, x)\, g_1(x))\leqslant 1+\varepsilon.$$

Then (3.30) and the monotonicity of $\lambda_n(\cdot, x)$ in the weight shows that for $n\geqslant n_1$ and $x\in J$,

$$\lambda_n(W^2g,x)/(\lambda_n(W^2,x)\, g(x))\leqslant(1+\varepsilon)\, \lambda_n(W^2g_1,x)/(\lambda_n(W^2,x)\, g_1(x))(g_1/g)(x)$$
$$\leqslant(1+\varepsilon)^2\,(1-\varepsilon)^{-1}.$$

Similarly, we obtain a lower bound. ∎

We shall briefly outline the proof of Theorem 1.6. Following is the analogue of Lemma 3.2:

LEMMA 3.5. *Assume the hypotheses of Theorem* 1.6. *Then for each x such that $|x|\leqslant 160q_n$, there exists polynomials $V_n(u)$ and $U_n(u)$ (depending on x) of degree at most C_1m_n, such that for $n=1, 2, 3, \ldots$,* (3.3) *and* (3.4) *hold uniformly for $|u|\leqslant 160q_n$. The constants C_1 and C_2 are independent of n and $|x|$, $|u|\leqslant 160q_n$, while $\theta=0.873$.*

Proof. Define h, V_n, and U_n by (3.5), (3,6), and (3.7), where we will choose $l(n)$ and A_n below. First (3.3) follows from (3.6) and (3.7). Next, for $|x|\leqslant 160q_n$,

$$|h'(x)|\leqslant|h'(0)|+\left|\int_0^x h''(u)\, du\right|$$
$$\leqslant Cq_n\max\{|(\log g)''(x)|: |x|\leqslant 160q_n\}$$
$$=Cq_nB_n,$$

say, and hence for $|x|, |u| \leqslant 160q_n$,

$$|h'(x)(u-x) \pm A_n(u-x)^2| \leqslant Cq_n^2(B_n + A_n).$$

Thus if

$$l(n) \geqslant 4Cq_n^2(B_n + A_n), \tag{3.31}$$

Lemma 3.1 and (3.5) show that uniformly for $|x|, |u| \leqslant 160q_n$, both (3.9) and (3.10) hold. Then (3.4) follows and the proof of the lemma is complete, provided we can choose A_n and $l(n)$ (independent of $|x| \leqslant 160q_n$) to satisfy (3.31) and

$$m_n \leqslant l(n) \leqslant Cm_n \tag{3.32}$$

and

$$|h(u) - h(x) - h'(x)(u-x)^2| \leqslant A_n(u-x)^2, \qquad |x|, |u| \leqslant 160q_n. \tag{3.33}$$

But there exists v between u and x such that

$$h(u) - h(x) - h'(x)(u-x)^2 = h''(v)(u-x)^2/2,$$

and so (3.33) is satisfied if

$$A_n \geqslant B_n/2 = \max\{|h''(v)|/2\colon |v| \leqslant 160q_n\}. \tag{3.34}$$

In view of the definition (1.23) of m_n, we can clearly choose A_n to satisfy (3.34) and $l(n)$ to satisfy (3.31) and (3.32). ∎

The proof of Theorem 1.6 is very similar to that of Theorem 1.4—merely substitute Lemma 3.5 for Lemma 3.2.

As discussed in the Introduction, Theorems 1.4 and 1.6 do not apply to $W(x) = W_2(x) = \exp(-x^2/2)$, since they require $q_n = o(\sqrt{n})$. Suppose now that g is positive and finite valued in $\mathbb{R}$, and that there exist polynomials $V_n(u)$ and $U_n(u)$ of degree l_n with

$$U_n^{-1}(x)(1 + O(\delta_n)) = g^{1/2}(x) = V_n(x)(1 + O(\delta_n)), \qquad |x| \leqslant 160q_n = 160\sqrt{n},$$

where

$$\lim_{n \to \infty} \delta_n = 0 \qquad \text{and} \qquad \lim_{n \to \infty} l_n/n = 0.$$

Proceeding along the lines of Theorem 1.4, we easily see that uniformly for $|x| \leqslant C_1\sqrt{n}$ (some C_1),

$$\lambda_n(W_2^2 g, x)/\{\lambda_n(W_2^2, x)\, g(x)\} = 1 + O(\delta_n) + O(l_n/n).$$

For example, if $g(x) = \exp(ax+b)$, we may choose $l_n \sim \sqrt{n}$ and $\delta_n = \theta_2^n$, some $0 < \theta_2 < 1$.

Note added in proof. Since completion of this paper in early 1985, there have been several relevant developments. For example, the estimates for Christoffel functions for $|x| \leqslant (1-\varepsilon)\, x_{n1}(W^2)$ alluded to after Theorem 1.6 have been established (D. Lubinsky and E. B. Saff, Uniform and mean approximation by certain weighted polynomials, *Constr. Approx.*, to appear). Hence, if (1.9) holds for $|x| \leqslant (1-\varepsilon)\, x_{n1}(W^2)$, any $0<\varepsilon<1$, so also do Theorems 1.2 to 1.6, rather than just for $|x| \leqslant Cq_n$, some C. In particular, this is valid for $W_m(x) = \exp(-x^m/2)$, $m = 2, 4, 6, \ldots$.

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